Trigonometry and Modelling - Answers

June 2017 Mathematics Advanced Paper 1: Pure Mathematics 3

1.

Question Number	Scheme	Marks
9(a)	$\sin 2x - \tan x = 2\sin x\cos x - \tan x$	M1
	$= \dfrac{2\sin x\cos^2 x}{\cos x} - \dfrac{\sin x}{\cos x}$	M1
	$= \dfrac{\sin x}{\cos x} \times (2\cos^2 x - 1)$	
	$= \tan x\cos 2x$	dM1 A1*
		(4)
(b)	$\tan x\cos 2x = 3\tan x\sin x \Rightarrow \tan x(\cos 2x - 3\sin x) = 0$	
	$\cos 2x - 3\sin x = 0$	M1
	$\Rightarrow 1 - 2\sin^2 x - 3\sin x = 0$	M1
	$\Rightarrow 2\sin^2 x + 3\sin x - 1 = 0 \Rightarrow \sin x = \dfrac{-3 \pm \sqrt{17}}{4} \Rightarrow x = \ldots$	M1
	Two of $x = 16.3°, 163.7°, 0, 180°$	A1
	All four of $x = 16.3°, 163.7°, 0, 180°$	A1
		(5)
		(9 marks)

(a)

M1 Uses a correct double angle identity involving $\sin 2x$ Accept $\sin(x+x) = \sin x\cos x + \cos x\sin x$

M1 **Uses** $\tan x = \dfrac{\sin x}{\cos x}$ with $\sin 2x = 2\sin x\cos x$ and attempts to combine the two terms using a common denominator. This can be awarded on two separate terms with a common denominator.
Alternatively uses $\sin x = \tan x\cos x$ and attempts to combine two terms using factorisation of $\tan x$

dM1 Both M's must have been scored. Uses a correct double angle identity involving $\cos 2x$.

A1* A fully correct solution with no errors or omissions. All notation must be correct and variables must be consistent
Withhold this mark if for instance they write $\tan x = \dfrac{\sin}{\cos}$

If the candidate $\times\cos x$ on line 1 and/or $\div\sin x$ they cannot score any more than one mark unless they are working with both sides of the equation or it is fully explained.

(b)

M1 The $\tan x$ must be cancelled or factorised out to produce $\cos 2x - 3\sin x = 0$ or $\dfrac{\cos 2x}{\sin x} = 3$ oe Condone slips

M1 Uses $\cos 2x = 1 - 2\sin^2 x$ to form a 3TQ=0 in $\sin x$ The = 0 may be implied by later work

M1 Uses the formula/completion of square or GC with invsin to produce at least one value for x
It may be implied by one correct value.
This mark **can** be scored from factorisation of their 3TQ in $\sin x$ **but only if** their quadratic factorises.

A1 Two of $x = 0, 180°, \text{awrt } 16.3°, \text{awrt } 163.7°$ or in radians two of awrt 0.28, 2.86, 0 and π or 3.14
This mark can be awarded as a SC for those students who just produce $0, 180°$ (or 0 and π) from $\tan x = 0$ or $\sin x = 0$.

A1 All four values in degrees $x = 0, 180°, \text{awrt } 16.3°, \text{awrt } 163.7°$ and no extra's inside the range $0 ,, x < 360°$.
Condone $0 = 0.0$ and $180° = 180.0°$ Ignore any roots outside range.

Alternatives to parts (a) and (b)

(a) Alt 1	$\tan x \cos 2x = \tan x\left(2\cos^2 x - 1\right)$	M1
	$= 2\tan x \cos^2 x - \tan x$	
	$= 2\dfrac{\sin x}{\cos x}\cos^2 x - \tan x$	M1
	$= 2\sin x \cos x - \tan x$	
	$= \sin 2x - \tan x$	dM1 A1
		(4)

a) Alt 1 Starting from the rhs

M1 Uses a correct double angle identity for $\cos 2x$. Accept any correct version including $\cos(x+x) = \cos x \cos x - \sin x \sin x$

M1 **Uses** $\tan x = \dfrac{\sin x}{\cos x}$ with $\cos 2x = 2\cos^2 x - 1$ and attempts to multiply out the bracket

dM1 Both M's must have been scored. It is for using $2\sin x \cos x = \sin 2x$

A1* A fully correct solution with no errors or omissions. All notation must be correct and variables must be consiste
See Main scheme for how to deal with candidates who $\div \tan x$

(a) Alt 2	$\sin 2x - \tan x \equiv \tan x \cos 2x$	
	$2\sin x \cos x - \tan x \equiv \tan x(2\cos^2 x - 1)$	M1
	$2\sin x \cos x - \cancel{\tan x} \equiv 2\tan x \cos^2 x - \cancel{\tan x}$	
	$2\sin x \cos x \equiv 2\dfrac{\sin x}{\cos x}\cos^2 x$	M1
	$2\sin x \cos x \equiv 2\sin x \cos x$	dM1
	+statement that it must be true	A1*

a) Alt 2 Candidates who use both sides

M1 Uses a correct double angle identity involving $\sin 2x$ **or** $\cos 2x$. Can be scored from either side
Accept $\sin(x+x) = \sin x \cos x + \cos x \sin x$ or $\cos(x+x) = \cos x \cos x - \sin x \sin x$

M1 **Uses** $\tan x = \dfrac{\sin x}{\cos x}$ with $\cos 2x = 2\cos^2 x - 1$ and cancels the $\tan x$ term from both sides

dM1 Uses a correct double angle identity involving $\sin 2x$ Both previous M's must have been scored

A1* A fully correct solution with no errors or omissions AND statement "hence true", "a tick", "QED". W[5]
All notation must be correct and variables must be consistent

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It is possible to solve part (b) without using the given identity. There are various ways of doing this, one of which is shown below.

$\sin 2x - \tan x = 3\tan x \sin x \Rightarrow \; 2\sin x\cos x - \dfrac{\sin x}{\cos x} = 3\dfrac{\sin x}{\cos x}\sin x$

$2\sin x\cos^2 x - \sin x = 3\sin^2 x$ — M1 Equation in $\sin x$ and $\cos x$

$2\sin x\left(1-\sin^2 x\right) - \sin x = 3\sin^2 x$ — M1 Equation in $\sin x$ only

$\left(2\sin^2 x + 3\sin x - 1\right)\sin x = 0$

$x = ..$ — M1 Solving equation to find at least one x

Two of $x = 16.3°, 163.7°, 0, 180°$ — A1

All four of $x = 16.3°, 163.7°, 0, 180°$ and no extras A1

Jan 2013 Mathematics Advanced Paper 1: Pure Mathematics 3

2.

Question Number	Scheme	Marks
6.	(i) $(\sin 22.5 + \cos 22.5)^2 = \sin^2 22.5 + \cos^2 22.5 + ...$	M1
	$= \sin^2 22.5 + \cos^2 22.5 + 2\sin 22.5\cos 22.5$	
	States or uses $\sin^2 22.5 + \cos^2 22.5 = 1$	B1
	Uses $2\sin x\cos x = \sin 2x \;\Rightarrow 2\sin 22.5\cos 22.5 = \sin 45$	M1
	$(\sin 22.5 + \cos 22.5)^2 = 1 + \sin 45$	A1
	$= 1 + \dfrac{\sqrt{2}}{2}$ or $1 + \dfrac{1}{\sqrt{2}}$ cso	A1
		(5)
	(ii) (a) $\cos 2\theta + \sin\theta = 1 \Rightarrow 1 - 2\sin^2\theta + \sin\theta = 1$	M1
	$\sin\theta - 2\sin^2\theta = 0$	
	$2\sin^2\theta - \sin\theta = 0$ or $k = 2$	A1*
		(2)
	(b) $\sin\theta(2\sin\theta - 1) = 0$	M1
	$\sin\theta = 0, \quad \sin\theta = \dfrac{1}{2}$	A1
	Any two of $0, 30, 150, 180$	B1
	All four answers $0, 30, 150, 180$	A1
		(4)
		(11 marks)

(i)	M1	Attempts to expand $(\sin 22.5+\cos 22.5)^2$. Award if you see $\sin^2 22.5+\cos^2 22.5+......$ There must be > two terms. Condone missing brackets ie $\sin 22.5^2+\cos 22.5^2+......$
	B1	Stating or using $\sin^2 22.5+\cos^2 22.5=1$. Accept $\sin 22.5^2+\cos 22.5^2=1$ as the intention is clear. Note that this may also come from using the double angle formula $\sin^2 22.5+\cos^2 22.5=(\frac{1-\cos 45}{2})+(\frac{1+\cos 45}{2})=1$
	M1	Uses $2\sin x\cos x=\sin 2x$ to write $2\sin 22.5\cos 22.5$ as sin 45 or sin(2×22.5)
	A1	Reaching the intermediate answer $1+\sin 45$
	A1	Cso $1+\frac{\sqrt{2}}{2}$ or $1+\frac{1}{\sqrt{2}}$. Be aware that both 1.707 and $\frac{2+\sqrt{2}}{2}$ can be found by using a calculator for 1+sin45. Neither can be accepted on their own without firstly seeing one of the two answers given above. **Each stage should be shown as required by the mark scheme.** Note that if the candidates use $(\sin\theta+\cos\theta)^2$ they can pick up the first M and B marks, but no others until they use $\theta=22.5$. All other marks then become available.
(iia)	M1	Substitutes $\cos 2\theta=1-2\sin^2\theta$ in $\cos 2\theta+\sin\theta=1$ to produce an equation in $\sin\theta$ only. It is acceptable to use $\cos 2\theta=2\cos^2\theta-1$ *or* $\cos^2\theta-\sin^2\theta$ as long as the $\cos^2\theta$ is subsequently replaced by $1-\sin^2\theta$
	A1*	Obtains the correct simplified equation in $\sin\theta$. $\sin\theta-2\sin^2\theta=0$ or $\sin\theta=2\sin^2\theta$ must be written in the form $2\sin^2\theta-\sin\theta=0$ as required by the question. Also accept $k=2$ as long as no incorrect working is seen.
(iib)	M1	Factorises or divides by $\sin\theta$. For this mark $1='k'\sin\theta$ is acceptable. If they have a 3 TQ in $\sin\theta$ this can be scored for correct factorisation
	A1	**Both** $\sin\theta=0$, and $\sin\theta=\frac{1}{2}$
	B1	Any two answers from 0, 30, 150, 180.
	A1	All four answers 0, 30, 150, 180 with no extra solutions inside the range. Ignore solutions outside the range.

Question Number	Scheme	Marks
6.alt 1	(i) $(\sin 22.5+\cos 22.5)^2=\sin^2 22.5+\cos^2 22.5+......$	M1
	$=\sin^2 22.5+\cos^2 22.5+2\sin 22.5\cos 22.5$	
	States or uses $\sin^2 22.5+\cos^2 22.5=1$	B1
	Uses $2\sin x\cos x=2\sqrt{\frac{1-\cos 2x}{2}}\sqrt{\frac{\cos 2x+1}{2}}\Rightarrow\sqrt{1-\cos 45}\sqrt{1+\cos 45}$	M1
	$=\sqrt{1-\cos^2 45}$	A1
	Hence $(\sin 22.5+\cos 22.5)^2=1+\frac{\sqrt{2}}{2}$ or $1+\frac{1}{\sqrt{2}}$	A1
		(5)

Question Number	Scheme	Marks
6.alt 2	(i) Uses Factor Formula $(\sin 22.5+\sin 67.5)^2 = \left(2\sin 45\cos 22.5\right)^2$	M1,A1
	Reaching the stage $=2\cos^2 22.5$	B1
	Uses the double angle formula $=2\cos^2 22.5 = 1+\cos 45$	M1
	$=1+\dfrac{\sqrt{2}}{2}$ or $1+\dfrac{1}{\sqrt{2}}$	A1
		(5)

Question Number	Scheme	Marks
6.alt 3	(i) Uses Factor Formula $(\cos 67.5+\cos 22.5)^2 = \left(2\cos 45\cos 22.5\right)^2$	M1,A1
	Reaching the stage $=2\cos^2 22.5$	B1
	Uses the double angle formula $=2\cos^2 22.5 = 1+\cos 45$	M1
	$=1+\dfrac{\sqrt{2}}{2}$ or $1+\dfrac{1}{\sqrt{2}}$	A1
		(5)

Jan 2012 Mathematics Advanced Paper 1: Pure Mathematics 3

3.

Question No	Scheme	Marks
8	(a) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}$	M1A1
	$= \frac{\frac{sinA}{cosA}+\frac{sinB}{cosB}}{1-\frac{sinAsinB}{cosAcosB}}$ $(\div\, cosAcosB)$	M1

		$= \dfrac{tanA + tanB}{1 - tanAtanB}$	A1 * (4)
	(b)	$\tan\left(\theta + \frac{\pi}{6}\right) = \dfrac{tan\theta + tan\frac{\pi}{6}}{1 - tan\theta tan\frac{\pi}{6}}$	M1
		$= \dfrac{tan\theta + \frac{1}{\sqrt{3}}}{1 - tan\theta\frac{1}{\sqrt{3}}}$	M1
		$= \dfrac{\sqrt{3}tan\theta + 1}{\sqrt{3} - tan\theta}$	A1 * (3)
	(c)	$\tan\left(\theta + \frac{\pi}{6}\right) = \tan(\pi - \theta)$.	M1
		$\left(\theta + \frac{\pi}{6}\right) = (\pi - \theta)$	dM1
		$\theta = \frac{5}{12}\pi$	ddM1 A1
		$\tan\left(\theta + \frac{\pi}{6}\right) = \tan(2\pi - \theta)$	dddM1
		$\theta = \frac{11}{12}\pi$	A1 (6)
			(13 MARKS)

(a)

M1 Uses the identity $\{\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}\} = \frac{sinAcosB \pm cosAsinB}{cosAcosB \mp sinAsinB}$. Accept incorrect signs for this. Just the right hand side is acceptable.

A1 Fully correct statement in terms of cos and sin $\{\tan(A+B)\} = \frac{sinAcosB + cosAsinB}{cosAcosB - sinAsinB}$

M1 Divide **both** numerator and denominator by cosAcosB. This can be stated or implied by working. If implied you must have seen at least one term modified on both the numerator and denominator.

A1* This is a given solution. The last two principal's reports have highlighted lack of evidence in such questions. Both sides of the identity must be seen or implied. Eg lhs=
The minimum expectation for full marks is

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinAcosB + cosAsinB}{cosAcosB - sinAsinB} = \frac{\frac{sinA}{cosA} + \frac{sinB}{cosB}}{1 - \frac{sinAsinB}{cosAcosB}} = \frac{tanA + tanB}{1 - tanAtanB}$$

The solution $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinAcosB + cosAsinB}{cosAcosB - sinAsinB} = \frac{tanA + tanB}{1 - tanAtanB}$ scores M1A1M0A0

The solution $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinAcosB + cosAsinB}{cosAcosB - sinAsinB}\ (\div\ cosAcosB) = \frac{tanA + tanB}{1 - tanAtanB}$ scores M1A1M1A0

(b)

M1 An attempt to use part (a) with A=θ and $B = \frac{\pi}{6}$. Seeing $\frac{tan\theta + tan\frac{\pi}{6}}{1 - tan\theta tan\frac{\pi}{6}}$ is enough evidence. Accept sign slips

M1 Uses the identity $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ *or* $\frac{\sqrt{3}}{3}$ in the rhs of the identity on both numerator and denominator

A1* cso. This is a given solution. Both sides of the identity must be seen. All steps must be correct with no unreasonable jumps. Accept

$$\tan\left(\theta + \frac{\pi}{6}\right) = \frac{tan\theta + tan\frac{\pi}{6}}{1 - tan\theta tan\frac{\pi}{6}} = \frac{tan\theta + \frac{1}{\sqrt{3}}}{1 - tan\theta\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}tan\theta + 1}{\sqrt{3} - tan\theta}$$

However the following is only worth 2 out of 3 as the last step is an unreasonable jump without further explanation

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$$\tan\left(\theta+\frac{\pi}{6}\right)=\frac{tan\theta+tan\frac{\pi}{6}}{1-tan\theta tan\frac{\pi}{6}}=\frac{tan\theta+\frac{\sqrt{3}}{3}}{1-tan\theta\frac{\sqrt{3}}{3}}=\frac{\sqrt{3}tan\theta+1}{\sqrt{3}-tan\theta}$$

(c)

M1 Use the given identity in (b) to obtain $\tan\left(\theta+\frac{\pi}{6}\right)=\tan(\pi-\theta)$. Accept sign slips

dM1 Writes down an equation that will give one value of θ , usually $\theta+\frac{\pi}{6}=\pi-\theta$. This is dependent upon the first M mark. Follow through on slips

ddM1 Attempts to solve their equation in θ. It must end θ= and the first two marks must have been scored.

A1 Cso θ=$\frac{5}{12}\pi$ or $\frac{11}{12}\pi$

dddM1 Writes down an equation that would produce a second value of θ . Usually $\theta+\frac{\pi}{6}=2\pi-\theta$

A1 cso θ= $\frac{5}{12}\pi$ $(accept\ \frac{\pi}{2.4})$and $\frac{11}{12}\pi$ with no extra solutions in the range. Ignore extra solutions outside the range.

Note that under this method one correct solution would score 4 marks. A small number of candidates find the second solution only. They would score 1,1,1,1,0,0

Alternative to (a) starting from rhs

M1 Uses correct identities for both *tan*A and *tan*B in the rhs expression. Accept only errors in signs

A1 $\frac{tanA+tanB}{1-tanAtanB}=\frac{\frac{sinA}{cosA}+\frac{sinB}{cosB}}{1-\frac{sinAsinB}{cosAcosB}}$

M1 Multiplies both numerator and denominator by *cos*A*cos*B. This can be stated or implied by working. If implied you must have seen at least one term modified on both the numerator and denominator

A1 This is a given answer. Correctly completes proof. All three expressions must be seen or implied.

$$\frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}=\frac{\sin(A+B)}{\cos(A+B)}=\tan(A+B)$$

Alternative to (a) starting from both sides

The usual method can be marked like this

M1 Uses correct identities for both *tan*A and *tan*B in the rhs expression. Accept only errors in signs

A1 $\frac{tanA+tanB}{1-tanAtanB}=\frac{\frac{sinA}{cosA}+\frac{sinB}{cosB}}{1-\frac{sinAsinB}{cosAcosB}}$

M1 Multiplies both numerator and denominator by *cos*A*cos*B. This can be stated or implied by working. If implied you must have seen at least one term modified on both the numerator and denominator

A1 Completes proof. Starting now from the lhs writes $\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}$

And then states that the lhs is equal to the rhs **Or** hence proven. There must be a statement of closure

Alternative to (b) from sin and cos

M1 Writes $\tan\left(\theta+\frac{\pi}{6}\right)=\frac{\sin\left(\theta+\frac{\pi}{6}\right)}{\cos\left(\theta+\frac{\pi}{6}\right)}=\frac{\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}}{\cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}}$

M1 Uses the identities $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$ and $cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ oe in the rhs of the identity on both numerator and denominator and divides both numerator and denominator by $\cos\theta$ to produce an identity in $\tan\theta$

A1 As in original scheme

Alternative solution for c. Starting with $1+\sqrt{3}tan\theta=(\sqrt{3}-\tan\theta)\,tan(\pi-\theta)$

Let $\tan\theta=t$

$$1+\sqrt{3}t=(\sqrt{3}-t)(-t)$$

$$t^2-2\sqrt{3}t-1=0$$

$$t=\frac{2\sqrt{3}\pm\sqrt{(12+4)}}{2}$$

$$=\sqrt{3}\pm2$$ Must find an exact surd

$$\theta=\frac{5\pi}{12},\ \frac{11\pi}{12}$$

Accept the use of a calculator for the A marks as long as there is an exact surd for the solution of the quadratic and exact answers are given.

M1 Starting with $1+\sqrt{3}tan\theta=(\sqrt{3}-\tan\theta)\,tan(\pi-\theta)$ expand $\tan(\pi-\theta)$ by the correct compound angle identity (or otherwise) and substitute $\tan\pi=0$ to produce an equation in $\tan\theta$

dM1 Collect terms and produce a 3 term quadratic in $\tan\theta$

ddM1 Correct use of quadratic formula to produce exact solutions to $\tan\theta$. All previous marks must have been scored.

dddM1 All 3 previous marks must have been scored. This is for producing two exact values for θ

A1 One solution $\frac{5}{12}\pi$ $(accept\ \frac{\pi}{2.4})$ or $\frac{11}{12}\pi$

A1 Both solutions $\frac{5}{12}\pi$ $(accept\ \frac{\pi}{2.4})$ and $\frac{11}{12}\pi$ and no extra solutions inside the range. Ignore extra solutions outside the range.

Special case: Watch for candidates who write $tan(\pi-\theta)=tan(\pi)-tan(\theta)=-tan(\theta)$ and proceed correctly. They will lose the first mark but potentially can score the others.

Solutions in degrees

Apply as before. Lose the first correct mark that would have been scored-usually 75^0

June 2011 Mathematics Advanced Paper 1: Pure Mathematics 3

4.

Question Number	Scheme	Marks
6 (a)	$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \frac{1-\cos 2\theta}{\sin 2\theta}$	M1
	$= \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$	M1A1
	$= \frac{\sin\theta}{\cos\theta} = \tan\theta$ cso	A1* (4)
(b)(i)	$\tan 15° = \frac{1}{\sin 30°} - \frac{\cos 30°}{\sin 30°}$	M1
	$\tan 15° = \frac{1}{\frac{1}{2}} - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}$ cso	dM1 A1* (3)

(b)(ii)	$\tan 2x = 1$	M1
	$2x = 45°$	A1
	$2x = 45° + 180°$	M1
	$x = 22.5°, 112.5°, 202.5°, 292.5°$	A1(any two) A1 (5)
		12 Marks
	Alt for (b)(i) $\tan 15° = \tan(60° - 45°)$ *or* $\tan(45° - 30°)$	
	$\tan 15° = \dfrac{\tan 60 - \tan 45}{1 + \tan 60 \tan 45}$ *or* $\dfrac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$	M1
	$\tan 15° = \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}}$ *or* $\dfrac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$	M1
	Rationalises to produce $\tan 15° = 2 - \sqrt{3}$	A1*